

Trapped Atomic Fermi Gases

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Abstract

A many-body system of fermion atoms with a model interaction characterized by the scattering length a is considered. We treat both a and the density as parameters assuming that the system can be created artificially in a trap. If a is negative the system becomes strongly correlated at densities $\rho \sim |a|^{-3}$, provided the scattering length is the dominant parameter of the problem. It means that we consider $|a|$ to be much bigger than the radius of the interaction or any other relevant parameter of the system. The density ρ_{c1} at which the compressibility vanishes is defined by $\rho_{c1} \sim |a|^{-3}$. Thus, a system composed of fermion atoms with the scattering length $a \rightarrow -\infty$ is completely unstable at low densities, inevitably collapsing until the repulsive core stops the density growth. As a result, any Fermi system possesses the equilibrium density and energy if the bare particle-particle interaction is sufficiently strong to make a negative and to be the dominant parameter. This behavior can be realized in a trap. Our results show that a low density neutron matter can have the equilibrium density.

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Although a theory of Fermi gases is not yet well developed, there are nevertheless new challenging experimental possibilities to explore trapped Fermi gases [1]. These are expected to stimulate a proper theoretical description of these Fermi quantum systems. Currently this description of quantum Fermi gases at their different states, which can be equilibrium, quasi-equilibrium or far from equilibrium, is usually based on tedious numerical calculations, particularly when the interaction between particles, or atoms, of a gas cannot be considered as weak. This implies that the dimensionless effective coupling constant of the interaction $p_F a \geq 1$. Here p_F is the Fermi momentum and is related to the system's density by $\rho = p_F^3/3\pi^2$, while a is the scattering length. There are a few cases, when it is possible to treat the properties analytically. The Random Phase Approximation (RPA) is applicable for a high density electron gas [2] and the low density approximation deals with dilute gases [3]. In both cases the kinetic energy T_k is assumed to be much bigger than the interaction energy E_{int} of the system. This permits the application of some kind of a perturbation theory. In the case of an electron liquid it turns out that the analytical RPA-like description is also possible not only at very high but also at medium densities when $T_k \sim E_{int}$ [4, 5]. Similar extension of the range of validity is impossible in the case of fermion systems at low densities ρ (there the gas approximation is not applicable if $p_F a \geq 1$, or $T_k \sim E_{int}$). If the pair interaction is attractive and sufficiently strong, the system can have a quasi-equilibrium or equilibrium states in which $T_k \simeq E_{int}$. On the other hand, these states are separated by special regions at which the incompressibility $K(\rho) \leq 0$ and the system is completely unstable. An experimental study performed on such Fermi-systems would be of great importance presenting new information on the behavior of dilute gases and on the gas-liquid phase transition. The observation of the existence of such regions and points at which $K(\rho) = 0$ can present a challenging problem for a theory designed to describe these peculiarities.

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In this Letter we address the above mentioned problem and consider an infinitely extended system composed of Fermi particles, or atoms, interacting by an artificially constructed potential with the desirable scattering length. We demonstrate that the consideration can be accomplished analytically provided that the pair interaction between fermions is characterized only by the scattering length. That is $|a|$ is much larger than the other relevant parameters of the interaction, or $a \rightarrow -\infty$. In this case one can say that the scattering length is the dominant parameter of the problem under consideration. As it will be demonstrated below, in such a case the system is located in the vicinity of the gas-liquid phase transition, transforming it into a strongly correlated one. Therefore, the problem of calculating its properties has to be treated for the most part qualitatively. Such an investigation is of great importance since it can be applied to fermion systems interacting via potentials with not only infinite, but also sufficiently large a . For instance, the scattering length a of neutron-neutron interaction is about -20 fm, thus greatly surpassing the radius of the interaction r_0 . Then this investigation can be viewed as the first step to study trapped Fermi gases, which are systems composed of Fermi atoms interacting by an artificially constructed potential with almost any desirable scattering length, similarly to that done for the trapped Bose gases, see e.g. [6].

Let us start by considering the general properties of a Fermi system with some attractive two-particle bare interaction $gV(r)$ of range r_0 and g defining the strength of the interaction. We assume that $gV(r)$ is sufficiently weak to create a two-particle bound state and that the scattering length a corresponding to this potential is the dominant parameter, being negative and finite, while $r_0/|a| \ll 1$. Then in the Hartree-Fock approximation the ground state energy $E_{HF}(\rho)$ is given by the following expression

$$E_{HF}(\rho) = \frac{3p_F^2}{10M}\rho + t_{HF}\rho^2, \quad (1)$$

where M is the particle mass, and the parameter t_{HF} , being negative, is entirely determined by the potential $gV(r)$. For instance, in the case of a short range δ -type interaction one has $t_{HF} = -g/4$. Eq. (1) shows that at low densities $E_{HF} > 0$ due to the kinetic energy term, but at sufficiently high densities $\rho \rightarrow \infty$ the potential energy $t_{HF}\rho^2$ becomes dominating, leading to the collapse of the system, with $E_{HF} \rightarrow -\infty$. Keeping in mind that the Hartree-Fock approximation gives the upper limit to the binding energy $E_{HF} \geq E$, one can conclude that the system does not have, in this case, an equilibrium density ρ_e and energy E_e since $E_e \rightarrow -\infty$ when $\rho \rightarrow \infty$ [8]. Note, that for a given and finite total number of particles N , the HF energy is not going to infinity and the system collapses into a small volume with the radius r_0 , with the density $\rho \sim N/r_0^3$. At the densities $\rho \rightarrow 0$ we deal with a dilute gas and the energy $E(\rho) \rightarrow 0$, remaining positive at these densities [7]. Therefore, it must have at least one maximum at the density ρ_m before it becomes negative, on the way to $E \rightarrow -\infty$. If the potential $gV(r)$ includes a kind of a "repulsive core" at sufficiently short distances, the system has an equilibrium density and energy, ρ_e and E_e , respectively, determined by the repulsive core strength and its radius $r_c \sim r_0$. The general features of the function $E(\rho)$ can be qualitatively represented by a simple expression

$$E(\rho) = \frac{3p_F^2}{10M}\rho + t_0\rho^2 + t_3\rho^3. \quad (2)$$

The first term of Eq. (2) is the kinetic energy T_k , while the second and the third terms are related to the interaction energy E_{int} determined by the potential $gV(r)$. The second term which is proportional to t_0 gives a qualitative description in the gas limit. The third term provides the behavior of $E(\rho)$ at higher densities, including that of the equilibrium density, so that $t_0 < 0$, and $0 < t_3$.

Now we apply Eq. (2) to demonstrate the most important features of the system under consideration:

a) when $\rho \rightarrow 0$ the third term on r.h.s. in Eq. (2) can be omitted. The kinetic energy is relatively very big, $T_k \gg E_{int}$, and $t_0 \sim a$, with $a < 0$ being the scattering length. In that case we have a dilute

Fermi gas with positive pressure P and incompressibility K , the latter being determined by the equation, see e.g. [9],

$$K(\rho) = \rho^2 \frac{dE^2(\rho)}{d\rho^2}. \quad (3)$$

b) on the way to higher densities, which can be achieved by applying an external pressure, the system reaches the density $\rho_{c1} < \rho_m$ at which the incompressibility is equal to zero, $K(\rho_{c1}) = 0$. Remembering that at the maximum the second derivative is negative, one can conclude, as it is seen from Eq. (3), that $K(\rho_m) < 0$. In the range $\rho_{c2} \geq \rho \geq \rho_{c1}$ the incompressibility is negative, $K < 0$, and as a result the system becomes totally unstable. In fact, in this density range all calculations of the ground state energy are meaningless since such a system cannot exist and thus be observed experimentally [9];

c) at some point $\rho = \rho_{c2} > \rho_m$ the contribution due to the repulsive core becomes sufficiently strong to prevent the further collapse of the system. The incompressibility attains $K = 0$ at $\rho_{c2} < \rho_e$, being positive at the higher densities. Finally, the system becomes stable at $\rho > \rho_{c2}$, reaching equilibrium density at ρ_e with equilibrium energy equal to E_e . Generally speaking, the system is quasi-stable at the densities $0 < \rho < \rho_{c1}$ and $\rho_{c2} < \rho < \rho_e$ because, being affected by a sufficiently strong external field, it occupies its real stable state with the energy E_e and density ρ_e [9]. For the sake of simplicity, we call these states stable states. It is obvious that $K(\rho_e) > 0$ being proportional to the second derivative at the minimum [see Eq. (3)]. It should be kept in mind that in this density domain, $\rho \geq \rho_{c2}$, the function $E(\rho)$ is determined by the repulsive part of the potential which makes $t_3 > 0$. As mentioned above, without this component of $gV(r)$ the system's energy would decrease, $E_{HF}(\rho) \rightarrow -\infty$, with the density growth, $\rho \rightarrow \infty$, thus inevitably collapsing. Indeed, in this case, the potential being pure attractive has no structure to ensure any stable states at the densities $\rho \geq \rho_{c1}$. As a result, one can write down a dimensionless expression for the ground state energy as a function of the only variable $z = p_F a$ [3, 7],

$$\alpha E(z) = z^5(1 + \beta(z)), \quad (4)$$

with $\alpha = 10\pi^2 M a^5$. Note, that the function $\beta(z)$ depends only on the variable z . In the low density limit, $|ap_F| \ll 1$ and when the interaction has the radius r_0 , Eq. (4) reads [7],

$$\alpha E(z) = z^5 \left[1 + \frac{10}{9\pi} z + \frac{4}{21\pi^2} (11 - 2 \ln 2) z^2 + \left(\frac{r_0}{a} \right)^3 z^3 \gamma \left(\frac{r_0}{a}, z \right) + \dots \right]. \quad (5)$$

Here the function $\gamma(y, z)$ is of the order of one, $\gamma(y, z) \sim 1$. It is seen from Eq. (5) that as soon as the scattering length becomes large enough, $|a| \gg r_0$, one can omit the contribution coming from the function γ and neglect all the term proportional to $(r_0/a)^3$. Then Eq. (5) reduces to Eq. (4). Thus, in the case $|a| \rightarrow \infty$ we can use Eq. (4) to determine the ground state energy E . Eq. (4) is valid up to the density ρ_{c1} which is a singular point of the function $\beta(z)$, since beyond this point $K < 0$, and the system is completely unstable. On the other hand, there is no physical reason to have another irregular point in the region $0 < \rho < \rho_{c1}$. Using Eq. (3) for the incompressibility and Eq. (4) for the energy, one can calculate the position of the point z_{c1} where $K = 0$. Denoting the corresponding z as $z_{c1} = c_0$, where c_0 is a dimensionless number, one is led to the conclusion that $\rho_{c1} \sim |a|^{-3}$ provided a is sufficiently large to be the only dominating parameter. As it is seen from Eq. (5), we can expect corrections of the order of $(r_0/a)^9$ to this universal behavior. Thus, the system has only one stable region at small densities $\rho \leq \rho_{c1}$ which decreases and even vanishes as soon as $a \rightarrow -\infty$, and the function β is in fact determined only in the region $|z| \leq |z_{c1}|$ [10].

Consider the behavior of the system when the density approaches $\rho \rightarrow \rho_{c1} \sim |a|^{-3}$ from the low density side. Normally, points ρ_{c1} and ρ_{c2} , overlooked in calculations because of the lack of self consistency [11],

which relates the linear response function of the system to its incompressibility K ,

$$\chi(q \rightarrow 0, i\omega \rightarrow 0) = - \left(\frac{d^2 E}{d\rho^2} \right)^{-1}. \quad (6)$$

These points can give important contributions to the ground state energy. To see this we express the energy of the system in the following form (see e.g. [11]),

$$E(\rho) = E_{HF}(\rho) - \frac{1}{2} \int [\chi(q, i\omega, g) - \chi_0(q, i\omega)] v(q) \frac{d\mathbf{q} d\omega dg}{g(2\pi)^4}, \quad (7)$$

where $\chi(q, i\omega, g)$ is the linear response function on the imaginary axis, $\chi_0(q, i\omega)$ is the linear response function of noninteracting particles, and $v(q)$ is the Fourier image of $gV(r)$. The integration over ω goes from $-\infty$ to $+\infty$, while the integration over the coupling constant g runs from zero to the real value of the coupling constant, i.e. to $g = 1$. At the point $\rho = \rho_{c1}$ the linear response function has a pole at the origin of coordinates $q = 0, \omega = 0$ due to Eq. (6). At the densities $\rho > \rho_{c1}$ the function $\chi(q, i\omega, g)$ has poles at finite values of the momentum q and frequencies $i\omega$. This prevents the integration over $i\omega$, making the integral in Eq. (7) divergent. Thus, we conclude that it is the contribution of these poles that reflects the system's instability in the density range $\rho_{c1} \leq \rho \leq \rho_{c2}$. Note, that violations of Eq. (6) lead to serious errors in the calculation of the ground state energy. Equation (7) can be rewritten, explicitly accounting for the effective interparticle interaction $R(q, i\omega, g)$, in the following form

$$E(\rho) = E_{HF}(\rho) - \frac{1}{2} \int \left[\frac{\chi_0^2(q, i\omega) R(q, i\omega, g)}{1 - R(q, i\omega, g) \chi_0(q, i\omega)} \right] v(q) \frac{d\mathbf{q} d\omega dg}{g(2\pi)^4}, \quad (8)$$

and χ is given by the following equation [4, 11]

$$\chi(q, \omega, g) = \frac{\chi_0(q, \omega)}{1 - R(q, \omega, g) \chi_0(q, \omega)}. \quad (9)$$

It is seen from Eqs. (6) and (9) that the denominator $(1 - R\chi_0)$ vanishes at $\rho \rightarrow \rho_{c1}$. As a result, we obtain

$$R(q \rightarrow 0, \omega = 0, g = 1) \propto \frac{1}{\chi_0(q, 0)} \sim \frac{1}{p_F M + q^2/p_F^2}. \quad (10)$$

Here $p_F^3 \sim \rho_{c1}$. If the scattering length tends to infinity driving $\rho_{c1} \rightarrow 0$, it follows from Eq. (10) that

$$R(q \rightarrow 0, 0, 1) \propto \frac{1}{q^2}, \quad (11)$$

and we can conclude that the effective interaction R behaves as a gravitational-like field, leading the system to collapse until the repulsive core stops the further squeezing of the system. Note, it is impossible to represent the denominator as a power series in $R\chi_0$ through approximating the expansion by a finite number of terms. This result is quite obvious since ρ_{c1} is a branch point in the function $E(\rho)$, which makes it impossible to expand that function in the vicinity of this point. Therefore, one should try to satisfy Eq. (6) in order to get proper results for the ground state calculations in the vicinity of the instability points [4, 5, 11].

We would like to demonstrate here the essential qualitative features of atoms in a trap, which can be observed experimentally at the densities near the critical density, $\rho \leq \rho_{c1}$. It follows from Eq. (9), at least

for small q and $\rho \rightarrow \rho_{c1}$, that the susceptibility $\chi(q \rightarrow 0, 0, 1)$, being proportional to the incompressibility (see Eq. (6)), is divergent. Therefore, the denominator in Eq. (9) can be expressed in the form

$$1 - R(q, 0, 1)\chi_0(q, 0) \simeq \alpha_0(\rho/\rho_{c1}) + \alpha_1 q^2/p_F^2. \quad (12)$$

Here $\alpha_0(\rho/\rho_{c1})$ is a positive function such that $\alpha_0(1) = 0$, and α_1 is a positive constant. As a result, the linear response function takes the form,

$$\chi(q \rightarrow 0, 0, 1)|_{\rho \rightarrow \rho_{c1}} \propto \frac{1}{q^2}. \quad (13)$$

Since the linear response function is the density-density correlation function, we obtain from Eq. (13) that the radius of the correlation tends to infinity as it must in the vicinity of the phase transition point [9]. For instance, the density fluctuations $\delta\rho(q)$ induced by an external potential, $\delta\rho(q) = \chi(q, 0)v_{ext}(q) \propto v_{ext}(q)/q^2$. Thus, an external field $v_{ext}(q)$ even it is of a short range cannot be screened and produces the fluctuations of the density of extremely long range. It seems that such a behavior can be realized in a trap through the observation of a strong light scattering by the density fluctuations of multi-fermion system when $\rho \rightarrow \rho_{c1}$. On the other hand, the same picture can be observed if the density is fixed at some ρ_0 value, while the critical density is driven to this density by adjusting the scattering length by an external field. If the bare potential were purely attractive, the interval of the densities $[0, \rho_{c1}]$ within which the system is stable, would vanish with the growth of $|a|$ since $\rho_{c1} \sim |a|^{-3}$. As a result, in the limit $a = -\infty$ the density $\rho_{c1} \rightarrow 0$, and the considered system would be completely unstable at any density. As soon as the scattering length deviates from its infinite value, that is $+\infty > a > -\infty$ the system comes back to its stable state at list in the range of the density values $\rho < \rho_{c1} \sim |a|^{-3}$. Note, as it follows from our consideration, that any Fermi system possesses an equilibrium density and energy if the bare particle-particle interaction contains a repulsive core and its attractive part is strong enough, so that $a \rightarrow -\infty$. Indeed, at sufficiently small densities the energy is negative. The system collapses (since at these densities the incompressibility $K \leq 0$) until the core stops the density growth. Therefore, the minimal value of the ground state energy must be negative when the repulsive core switches on to prevent the system from further collapse.

Note that trapped alkali Fermi gases are spin polarized and cannot interact in s-wave channel. Consider a two component system realized by mixtures of two Fermi alkali gases, say ^6Li and ^{40}K . We assume that the corresponding number densities ρ_1 and ρ_2 of these gases coincide, $\rho_1 = \rho_2 = \rho$. In that case, we come back to our case: at each level there are two spin-polarized atoms interacting in s-wave and with "effective" mass $M^* = M_1 M_2 / (M_1 + M_2)$. Here M_1 and M_2 are the particle masses of the alkali gases under consideration. As a result upon using Eq. (4), we obtain $\rho_{c1} \sim |a_c|^{-3}$, where a_c is the scattering length related to the interaction between the different alkali atoms. It is possible to consider a large variety of two component Fermi systems with $\rho_1 \neq \rho_2$, or even two component systems composed of Fermi and Bose gases which can retain the significant features of the considered one component Fermi system. A detailed investigation of such systems is in progress and will be published elsewhere. It is worth remarking, that superfluid correlation cannot stop the system from squeezing, since their contribution to the ground state energy is negative. After all, the superfluid correlation can be considered as additional degrees of freedom which can therefore only decrease the energy. Recent considerations show that the compressibility related to the superfluid correlation is positive (see, e.g. [13] and references therein). Therefore we do not expect that the contribution coming from the superfluid correlation can lead to a significant change in our estimate of the critical density, $\rho_{c1} \sim |a|^{-3}$. While deriving the exact relationship, one has to include a proper treatment of the pairing.

A liquid similar to the model considered above exists in Nature, viz. liquid ^3He . If a $^3\text{He}_2$ dimer exists, obviously its bounding energy does not exceed the bounding energy of a $^4\text{He}_2$ dimer which is 10^{-4} meV [14]. The ground state energy of liquid helium is about $2 \cdot 10^{-1}$ meV per atom. Because of this huge difference

in binding energies, it is evident that the contribution coming from the binding energy of the dimer to the ground state energy of the liquid is insignificant. In fact, numerical calculations show that the pair potential is rather weak to produce the dimer ${}^3\text{He}_2$ [15]. Thus, one can reliably consider an infinite homogeneous system of Helium atoms as consisting of particles interacting via pair potentials, characterized by a very large but finite scattering length $|a| \gg r_0$. The following additional remark is appropriate. It seems quite probable that the neutron-neutron scattering length ($a \simeq -20$ fm) is sufficiently large to permit the neutron matter to have an equilibrium energy and density [12].

In summary, a system of fermions interacting by an artificial potential has been considered. The qualitative consideration presented above gives strong evidence that the system becomes unstable at densities $\rho \geq \rho_{c1}$. In the vicinity of $\rho_{c1} \sim |a|^{-3}$, there are long-ranged fluctuations of the density which can be realized in a trap through the strong scattering of light. Our results suggest that the equation of state of a low density neutron matter has peculiarities.

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